COMMON FIXED POINT THEOREMS FOR TWO MAPPINGS IN $\mathcal{M}$-FUZZY METRIC SPACES

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Abstract

In this paper, we prove some common fixed point theorems for two nonlinear mappings in complete $\mathcal{M}$-fuzzy metric spaces. Our main results improved versions of several fixed point theorems in complete fuzzy metric spaces.

1. Introduction

The concept of *fuzzy sets* was introduced initially by Zadeh [27] in 1965. Since then, to apply this concept in topology and analysis, many authors [9, 17, 19, 24] have expansively developed the theory of fuzzy sets and application. George and Veeramani [8] and Kramosil and Michalek [11] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and E-infinity theory which were given and studied by El-Naschie [3-6]. Many authors [7, 10, 12, 18, 20, 23] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [25] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki [25] proved fuzzy common fixed point theorem by a strong definition of a Cauchy sequence (see Note 3.13 and Definition 3.15 of [8], also [23, 26]).

On the other hand, Dhage [1, 2] introduced the notion of generalized metric or D-metric spaces and claimed that D-metric convergence defines a Hausdorff topology and D-metric is sequentially continuous in all the three variables. Many authors have used these claims in proving fixed point theorems in D-metric spaces, but, unfortunately, almost all theorems in D-metric spaces are not valid (see [13-16, 22]).

Recently, Sedghi et al. [21] introduced D^*-metric which is a probable modification of the definition of D-metric introduced by Dhage [1, 2] and proved some basic properties in D^*-metric spaces. Also, using the concept of the D^*-metrics, they defined $\mathcal{M}$-fuzzy metric space and proved some related fixed point theorems for some nonlinear mappings in complete $\mathcal{M}$-fuzzy metric spaces.

In this paper, we prove some common fixed point theorems for two nonlinear mappings in complete $\mathcal{M}$-fuzzy metric spaces. Our main results improved versions of several fixed point theorems in complete fuzzy metric spaces.

In what follows (X, D^*) will denote a D^*-metric space, N the set of all natural numbers and R^+ the set of all positive real numbers.

Definition 1.1. ([21]) Let X be a nonempty set. A *generalized metric* (or *D^*-metric*) on X is a function: $D^* : X^3 \longrightarrow R^+$ that satisfies the following conditions: for any $x, y, z, a \in X$,

(1) $D^*(x, y, z) \geq 0$,
(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$ (symmetry), where p is a permutation function,
(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair (X, D^*) is called a *generalized metric space* (or *D^*-metric space.*

Some immediate examples of such a function are as follows:

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$.
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, where d is the ordinary metric on X.
(c) If $X = R^n$, then we define

$$D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{\frac{1}{p}}$$

for any $p \in R^+$.

(d) If $X = R^+$, then we define

$$D^*(x, y, z) = \begin{cases} 0, & if x = y = z, \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

In a D^*-metric space (X, D^*), we can prove that $D^*(x, x, y) = D^*(x, y, y)$.

Let (X, D^*) be a D^*-metric space. For any $r > 0$, define the open ball with the center x and radius r as follows:

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}.$$

Example 1.2. Let $X = R$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$. Thus we have

$$\begin{aligned} B_{D^*}(1, 2) &= \{y \in R : D^*(1, y, y) < 2\} \\ &= \{y \in R : |y - 1| + |y - 1| < 2\} \\ &= \{y \in R : |y - 1| < 1\} \\ &= (0, 2). \end{aligned}$$

Definition 1.3. ([21]) Let (X, D^*) be a D^*-metric space and $A \subset X$.

(1) If, for any $x \in A$, there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$, then A is called an *open subset* of X.
(2) A subset A of X is said to be D^*-*bounded* if there exists $r > 0$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.
(3) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ if

$$D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0 \quad (n \to \infty).$$

That is, for any $\epsilon > 0$, there exists $n_0 \in N$ such that

$$D^*(x, x, x_n) < \epsilon, \quad \forall n \geq n_0.$$

Equivalently, for any $\epsilon > 0$, there exists $n_0 \in N$ such that

$$D^*(x, x_n, x_m) < \epsilon, \quad \forall n, m \geq n_0.$$

(4) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for any $\epsilon > 0$, there exits $n_0 \in N$ such that

$$D^*(x_n, x_n, x_m) < \epsilon, \quad \forall n, m \geq n_0.$$

(5) A D^*-metric space (X, D^*) is said to be *complete* if every Cauchy sequence is convergent.

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$. Then τ is a *topology* on X (induced by the D^*-metric D^*).

Definition 1.4. ([21]) Let (X, D^*) be a D^*-metric space. D^* is said to be *continuous function* on $X^3 \times (0, \infty)$ if

$$\lim_{n \to \infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$$

whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$, i.e.,

$$\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z.$$

Remark 1.5. ([21]) (1) Let (X, D^*) be a D^*-metric space. Then D^* is continuous function on X^3.

(2) If a sequence $\{x_n\}$ in X converges to a point $x \in X$, then the limit x is unique.

(3) If a sequence $\{x_n\}$ in X is converges to a point x, then $\{x_n\}$ is a Cauchy sequence in X.

Recently, motivated by the concept of D^*-metrics, Sedghi et al. [21] introduced the concept of $\mathcal{M}$-fuzzy metric spaces and their properties and, further, proved some related common fixed theorems for some contractive type mappings in $\mathcal{M}$-fuzzy metric spaces.

Definition 1.6. ([21]) A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a *continuous t-norm* if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0, 1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t-norm are $a*b = ab$ and $a*b = \min(a, b)$.

Definition 1.7. ([21]) A 3-tuple $(X, \mathcal{M}, *)$ is called an *$\mathcal{M}$-fuzzy metric space* if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z, a \in X$ and $t, s > 0$,

(1) $\mathcal{M}(x, y, z, t) > 0$,
(2) $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,
(3) $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$ (symmetry), where p is a permutation function,
(4) $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s)$,
(5) $\mathcal{M}(x, y, z, t) : X^3 \times (0, \infty) \to [0, 1]$ is continuous with respect to t.

Remark 1.8. Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space. Then, for any $t > 0$, $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space. For any $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with the center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}.$$

A subset A of X is called an open set if, for all $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

Definition 1.9. ([21]) Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\mathcal{M}(x, x, x_n, t) \to 1$ as $n \longrightarrow \infty$ for any $t > 0$.
(2) A sequence $\{x_n\}$ is called a Cauchy sequence if, for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in N$ such that
$$\mathcal{M}(x_n, x_n, x_m, t) > 1 - \epsilon, \quad \forall n, m \geq n_0.$$
(3) An $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ is said to be complete if every Cauchy sequence in X is convergent.

Example 1.10. Let X is a nonempty set and D^* be the D^*-metric on X. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. For any $t \in]0, \infty[$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}, \quad \forall x, y, z \in X.$$

It is easy to see that $(X, \mathcal{M}, *)$ is a $\mathcal{M}$-fuzzy metric space.

Remark 1.11. Let $(X, M, *)$ is a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \longrightarrow [0, 1]$ by

$$\mathcal{M}(x, y, z, t) = M(x, y, t) * M(y, z, t) * M(z, x, t), \quad \forall x, y, z \in X,$$

then $(X, \mathcal{M}, *)$ is an $\mathcal{M}$-fuzzy metric space.

Lemma 1.12. ([21]) *Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space. Then, for all $x, y, z \in X$ and $t > 0$, $\mathcal{M}(x, y, z, t)$ is nondecreasing with respect to t.*

Definition 1.13. ([21]) Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous function on $X^3 \times (0, \infty)$ if

$$\lim_{n\to\infty} \mathcal{M}(x_n, y_n, z_n, t_n) = \mathcal{M}(x, y, z, t)$$

whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z, t) \in X^3 \times (0, \infty)$, that is,

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} y_n = y, \quad \lim_{n\to\infty} z_n = z,$$

$$\lim_{n\to\infty} \mathcal{M}(x, y, z, t_n) = \mathcal{M}(x, y, z, t).$$

Lemma 1.14. ([21]) *Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $X^3 \times (0, \infty)$.*

Lemma 1.15. ([21]) *Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space. If we define $E_{\lambda,\mathcal{M}} : X^3 \to R^+ \cup \{0\}$ by*

$$E_{\lambda,\mathcal{M}}(x, y, z) = \inf\{t > 0 \ : \ \mathcal{M}(x, y, z, t) > 1 - \lambda\}, \quad \forall \lambda \in (0, 1),$$

then we have the following:

(1) *For any $\mu \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that*

$$\begin{aligned} &E_{\mu,\mathcal{M}}(x_1, x_1, x_n) \\ &\le E_{\lambda,\mathcal{M}}(x_1, x_1, x_2) + E_{\lambda,\mathcal{M}}(x_2, x_2, x_3) + \cdots + E_{\lambda,\mathcal{M}}(x_{n-1}, x_{n-1}, x_n) \end{aligned}$$

for any $x_1, x_2, \cdots, x_n \in X$.

(2) *A sequence $\{x_n\}$ is convergent in an $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ if and only if $E_{\lambda,\mathcal{M}}(x_n, x_n, x) \to 0$. Also, the sequence $\{x_n\}$ is a Cauchy sequence in X if and only if it is a Cauchy sequence with $E_{\lambda,\mathcal{M}}$.*

Lemma 1.16. ([21]) *Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space. If there exists $k > 1$ such that*

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \ge \mathcal{M}(x_0, x_0, x_1, k^n t), \quad \forall n \ge 1,$$

then $\{x_n\}$ is a Cauchy sequence in X.

Definition 1.17. ([7]) We say that an $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, *)$ has the property (C) if it satisfies the following condition: For some $x, y, z \in X$,

$$\mathcal{M}(x, y, z, t) = C, \quad \forall t > 0, \quad \Longrightarrow \quad C = 1.$$

2. The main results

Now, we are ready to give main results in this paper.

Theorem 2.1. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space and S, T be two self-mappings of X satisfying the following conditions:*

(i) *there exists a constant $k \in (0, 1)$ such that*

$$\mathcal{M}(Sx, TSx, Ty, kt) \geq \gamma(\mathcal{M}(x, Sx, y, t)), \quad \forall x, y \in X, \tag{2.1}$$

or

$$\mathcal{M}(Ty, STy, Sx, kt) \geq \gamma(\mathcal{M}(y, Ty, x, t)), \quad \forall x, y \in X, \tag{2.2}$$

where $\gamma : [0, 1] \to [0, 1]$ is a function such that $\gamma(a) \geq a$ for all $a \in [0, 1]$,

(ii) *$ST = TS$.*

If $(X, \mathcal{M}, *)$ have the property (C), then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X, define

$$\begin{cases} x_{2n+1} = Tx_{2n}, \\ x_{2n+2} = Sx_{2n+1}, \quad \forall n \geq 0. \end{cases} \tag{2.3}$$

(1) Let $d_m(t) = \mathcal{M}(x_m, x_{m+1}, x_{m+1}, t)$ for any $t > 0$. Then, for any even $m = 2n \in N$, by (2.1) and (2.3), we have

$$\begin{aligned} d_{2n}(kt) &= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, kt) \\ &= \mathcal{M}(Sx_{2n-1}, Tx_{2n}, Tx_{2n}, kt) \\ &= \mathcal{M}(Sx_{2n-1}, TSx_{2n-1}, Tx_{2n}, kt) \\ &\geq \gamma(\mathcal{M}(x_{2n-1}, Sx_{2n-1}, x_{2n}, t)) \\ &\geq \mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t) \\ &= d_{2n-1}(t). \end{aligned}$$

Thus $d_{2n}(kt) \geq d_{2n-1}(t)$ for all even $m = 2n \in N$ and $t > 0$.

Similarly, for any odd $m = 2n + 1 \in N$, we have also

$$d_{2n+1}(kt) \geq d_{2n}(t).$$

Hence we have

$$d_n(kt) \geq d_{n-1}(t), \quad \forall n \geq 1. \tag{2.4}$$

Thus, by (2.4), we have

$$\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_n, x_n, \frac{1}{k}t) \geq \cdots \geq \mathcal{M}(x_0, x_1, x_1, \frac{1}{k^n}t).$$

Therefore, by Lemma 1.16, $\{x_n\}$ is a Cauchy sequence in X and, by the completeness of X, $\{x_n\}$ converges to a point x in X and so

$$\begin{aligned} \lim_{n\to\infty} x_{2n+1} &= \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Sx_{2n+1} \\ &= \lim_{n\to\infty} x_{2n+2} = x. \end{aligned}$$

Now, we prove that $Tx = x$. Replacing x, y by x_{2n-1}, x, respectively, in (i), we obtain

$$\mathcal{M}(Sx_{2n-1}, TSx_{2n-1}, Tx, kt) \geq \gamma(\mathcal{M}(x_{2n-1}, Sx_{2n-1}, x, t)),$$

that is,

$$\begin{aligned} \mathcal{M}(x_{2n}, x_{2n+1}, Tx, kt) &\geq \gamma(\mathcal{M}(x_{2n-1}, x_{2n}, x, t)) \\ &\geq \mathcal{M}(x_{2n-1}, x_{2n}, x, t). \end{aligned} \tag{2.5}$$

Letting $n \longrightarrow \infty$ in (2.5), we have

$$\mathcal{M}(x, x, Tx, kt) \geq \mathcal{M}(x, x, x, t) = 1,$$

which implies that $Tx = x$, that is, x is a fixed point of T.

Next, we prove that $Sx = x$. Replacing x, y by x, x_{2n}, respectively, in (2.1), we obtain

$$\mathcal{M}(Sx, TSx, Tx_{2n}, kt) \geq \gamma(\mathcal{M}(x, Sx, x_{2n}, t)) \geq \mathcal{M}(x, Sx, x_{2n}, t).$$

By (ii), since $TS = ST$, we get

$$\mathcal{M}(Sx, Sx, Tx_{2n}, kt) \geq \gamma(\mathcal{M}(x, Sx, x_{2n}, t)) \geq \mathcal{M}(x, Sx, x_{2n}, t). \tag{2.6}$$

Letting $n \longrightarrow \infty$ in (2.6), we have

$$\mathcal{M}(Sx, Sx, x, kt) \geq \mathcal{M}(x, Sx, x, t)$$

and hence

$$\begin{aligned} \mathcal{M}(x, Sx, x, t) &\geq \mathcal{M}(x, Sx, x, \frac{1}{k}t) \\ &\geq \mathcal{M}(x, Sx, x, \frac{1}{k^2}t) \\ &\quad \cdots \\ &\geq \mathcal{M}(x, Sx, x, \frac{1}{k^n}t). \end{aligned}$$

On the other hand, it follows from Lemma 1.12 that

$$\mathcal{M}(x, Sx, x, k^n t) \leq \mathcal{M}(x, Sx, x, t).$$

Hence $\mathcal{M}(x, Sx, x, t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, *)$ has the property (C), it follows that $C = 1$ and so $Sx = x$, that is, x is a fixed point of S. Therefore, x is a common fixed point of the self-mappings S and T.

(2) By using (2.2) and (2.3), let $d_m(t) = \mathcal{M}(x_{m+1}, x_m, x_m, t)$ for any $t > 0$. Then, for any even $m = 2n \in N$, we have

$$\begin{aligned} d_{2n}(kt) &= \mathcal{M}(x_{2n+1}, x_{2n}, x_{2n}, kt) \\ &= \mathcal{M}(Tx_{2n}, Sx_{2n-1}, Sx_{2n-1}, kt) \\ &= \mathcal{M}(Tx_{2n}, STx_{2n-2}, Sx_{2n-1}, kt) \\ &\geq \gamma(\mathcal{M}(x_{2n}, Tx_{2n-2}, x_{2n-1}, t)) \\ &\geq \mathcal{M}(x_{2n}, Tx_{2n-2}, x_{2n-1}, t) \\ &\geq \mathcal{M}(x_{2n}, x_{2n-1}, x_{2n-1}, t) \\ &= d_{2n-1}(t). \end{aligned}$$

Thus $d_{2n}(kt) \geq d_{2n-1}(t)$ for all even $m = 2n \in N$ and $t > 0$.

Similarly, for any odd $m = 2n + 1 \in N$, we have also

$$d_{2n+1}(kt) \geq d_{2n}(t).$$

Hence we have

$$d_n(kt) \geq d_{n-1}(t), \quad \forall n \geq 1.$$

The remains of the proof are almost same to the case of (2.1).

Now, to prove the uniqueness, let x' be another common fixed point of S and T. Then we have

$$\begin{aligned} \mathcal{M}(x, x, x', kt) &= \mathcal{M}(Sx, TSx, Tx', kt) \\ &\geq \gamma(\mathcal{M}(x, Sx, x', t)) \\ &\geq \mathcal{M}(x, x, x', t), \end{aligned}$$

which implies that

$$\begin{aligned} \mathcal{M}(x, x, x', t) &\geq \mathcal{M}(x, x, x', \frac{1}{k}t) \\ &\geq \mathcal{M}(x, x, x', \frac{1}{k^2}t) \\ &\cdots \\ &\geq \mathcal{M}(x, x, x', \frac{1}{k^n}t). \end{aligned}$$

On the other hand, it follows from Lemma 2.12 that

$$\mathcal{M}(x, x, x', t) \leq \mathcal{M}(x, x, x', \frac{1}{k^n}t)$$

and hence $\mathcal{M}(x, x, x', t) = C$ for all $t > 0$. Since $(X, \mathcal{M}, *)$ has the property (C), it follows that $C = 1$, that is, $x = x'$. Therefore, x is a unique common fixed point of S and T. This completes the proof. □

By Theorem 2.1, we have the following:

Corollary 2.2. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space. Let T be a mapping from X into itself such that there exists a constant $k \in (0,1)$ such that*

$$\mathcal{M}(Tx, T^2x, Ty, kt) \geq \mathcal{M}(x, Tx, y, t), \quad \forall x, y \in X.$$

If $(X, \mathcal{M}, *)$ have the property (C), then T have a unique fixed point in X.

Proof. By Theorem 2.1, if we set $\gamma(a) = a$ and $S = T$, then the conclusion follows. □

Corollary 2.3. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space. Let T be a mapping from X into itself such that there exists a constant $k \in (0,1)$ such that*

$$\mathcal{M}(T^n x, T^{2n}x, T^n y, kt) \geq \mathcal{M}(x, T^n x, y, t)$$

for all $x, y \in X$ and $n \geq 2$. If $(X, \mathcal{M}, *)$ has the property (C), then T have a unique fixed point in X.

Proof. By Corollary 2.2, T^n have a unique fixed point in X. Thus there exists $x \in X$ such that $T^n x = x$. Since

$$T^{n+1}x = T^n(Tx) = T(T^n x) = Tx,$$

we have $Tx = x$. □

Next, by using Lemma 1.16 and the property (C), we can prove the main results in this paper.

Theorem 2.4. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let S and T be mappings from X into itself such that there exists a constant $k \in (0,1)$ such that*

$$\begin{aligned} &\mathcal{M}(Sx, Ty, Ty, kt) \\ &\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(y, Ty, Ty, t) \\ &\quad + c(t)\mathcal{M}(x, Ty, Ty, \alpha t) + d(t)\mathcal{M}(y, Sx, Sx, (2-\alpha)t) \\ &\quad + e(t)\mathcal{M}(x, y, y, t) \end{aligned} \tag{2.7}$$

for all $x, y \in X$ and $\alpha \in (0,2)$, where $a, b, c, d, e : [0, \infty) \longrightarrow [0,1]$ are five functions such that

$$a(t) + b(t) + c(t) + d(t) + e(t) = 1, \quad \forall t \in [0, \infty).$$

Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Then there exist $x_1, x_2 \in X$ such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Inductively, we can construct a sequence $\{x_n\}$ in X such that

$$\begin{cases} x_{2n+1} = Sx_{2n}, \\ x_{2n+2} = Tx_{2n+1}, \quad \forall n \geq 0. \end{cases} \tag{2.8}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. If we set

$$d_m(t) = \mathcal{M}(x_m, x_{m+1}, x_{m+1}, t), \quad \forall t > 0, \tag{2.9}$$

then we prove that $\{d_m(t)\}$ is increasing with respect to $m \in N$. In fact, for any odd $m = 2n+1 \in N$, we have

$$\begin{aligned}
&d_{2n+1}(kt)\\
&= \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, kt)\\
&= \mathcal{M}(Sx_{2n}, Tx_{2n+1}, Tx_{2n+1}, kt)\\
&\geq a(t)\mathcal{M}(x_{2n}, Sx_{2n}, Sx_{2n}, t) + b(t)\mathcal{M}(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t)\\
&\quad + c(t)\mathcal{M}(x_{2n}, Tx_{2n+1}, Tx_{2n+1}, \alpha t)\\
&\quad + d(t)\mathcal{M}(x_{2n+1}, Sx_{2n}, Sx_{2n}, (2-\alpha)t)\\
&\quad + e(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)\\
&= a(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t) + b(t)\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t)\\
&\quad + c(t)\mathcal{M}(x_{2n}, x_{2n+2}, x_{2n+2}, \alpha t)\\
&\quad + d(t)\mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+1}, (2-\alpha)t)\\
&\quad + e(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)
\end{aligned}$$

and so

$$\begin{aligned}
&d_{2n+1}(kt)\\
&\geq a(t)d_{2n}(t) + b(t)d_{2n+1}(t) + c(t)d_{2n}(t) * d_{2n+1}(qt)\\
&\quad + d(t) + e(t)d_{2n}(t).
\end{aligned} \tag{2.10}$$

The equality in (2.10) is true because, if set $\alpha = 1+q$ for any $q \in (k,1)$, then

$$\begin{aligned}
&\mathcal{M}(x_{2n}, x_{2n+2}, x_{2n+2}, (1+q)t)\\
&= \mathcal{M}(x_{2n}, x_{2n}, x_{2n+2}, (1+q)t)\\
&\geq \mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t) * \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, qt)\\
&= d_{2n}(t) * d_{2n+1}(qt).
\end{aligned}$$

Now, we claim that

$$d_{2n+1}(t) \geq d_{2n}(t), \quad \forall n \geq 1.$$

In fact, if $d_{2n+1}(t) < d_{2n}(t)$, then, since

$$d_{2n+1}(qt) * d_{2n}(t) \geq d_{2n+1}(qt) * d_{2n+1}(qt) = d_{2n+1}(qt)$$

in (3.10), we have

$$\begin{aligned}
d_{2n+1}(kt) &> a(t)d_{2n+1}(qt) + b(t)d_{2n+1}(qt) + c(t)d_{2n+1}(qt)\\
&\quad + d(t)d_{2n+1}(qt) + e(t)d_{2n+1}(qt)\\
&= d_{2n+1}(qt)
\end{aligned}$$

and so $d_{2n+1}(kt) > d_{2n+1}(qt)$, which is a contradiction. Hence $d_{2n+1}(t) \geq d_{2n}(t)$ for all $n \in N$ and $t > 0$. By (2.10), we have

$$\begin{aligned} d_{2n+1}(kt) &\geq a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)d_{2n}(qt) * d_{2n}(qt) \\ &\quad + d(t)d_{2n}(qt) + ep(t)d_{2n}(qt) \\ &= d_{2n}(qt). \end{aligned}$$

Now, if $m = 2n$, then, by (2.9), we have

$$\begin{aligned} &d_{2n}(kt) \\ &= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, kt) \\ &= \mathcal{M}(Sx_{2n-1}, Tx_{2n}, Tx_{2n}, kt) \\ &\geq a(t)\mathcal{M}(x_{2n-1}, Sx_{2n-1}, Sx_{2n-1}, t) + b(t)\mathcal{M}(x_{2n}, Tx_{2n}, Tx_{2n}, t) \\ &\quad + c(t)\mathcal{M}(x_{2n-1}, Tx_{2n}, Tx_{2n}, \alpha t) \\ &\quad + d(t)\mathcal{M}(x_{2n}, Sx_{2n-1}, Sx_{2n-1}, (2-\alpha)t) \\ &\quad + e(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t) \\ &= a(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t) + b(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t) \\ &\quad + c(t)\mathcal{M}(x_{2n-1}, x_{2n+1}, x_{2n+1}, \alpha t) + d(t)\mathcal{M}(x_{2n}, x_{2n}, x_{2n}, (2-\alpha)t) \\ &\quad + e(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t) \end{aligned}$$

and so

$$\begin{aligned} d_{2n}(kt) &\geq a(t)d_{2n-1}(t) + b(t)d_{2n}(t) + c(t)d_{2n-1}(t) * d_{2n}(qt) \\ &\quad + d(t) + e(t)d_{2n-1}(t). \end{aligned} \tag{2.11}$$

The equality in (2.11) is true because, if $\alpha = 1 + q$ for any $q \in (k, 1)$, then

$$\begin{aligned} &\mathcal{M}(x_{2n-1}, x_{2n+1}, x_{2n+1}, (1+q)t) \\ &= \mathcal{M}(x_{2n-1}, x_{2n-1}, x_{2n+1}, (1+q)t) \\ &\geq \mathcal{M}(x_{2n-1}, x_{2n-1}, x_{2n}, t) * \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, qt) \\ &= d_{2n-1}(t) * d_{2n}(qt). \end{aligned}$$

Now, we also claim that

$$d_{2n}(t) \geq d_{2n-1}(t), \quad \forall n \geq 1.$$

In fact, if $d_{2n}(t) < d_{2n-1}(t)$, then, since

$$d_{2n}(qt) * d_{2n-1}(t) \geq d_{2n}(qt) * d_{2n}(qt) = d_{2n}(qt)$$

in (3.11), we have

$$\begin{aligned} &d_{2n}(kt) \\ &> a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)d_{2n}(qt) + d(t)d_{2n}(qt) + e(t)d_{2n}(qt) \\ &= d_{2n}(qt) \end{aligned}$$

and so $d_{2n}(kt) > d_{2n}(qt)$, which is a contradiction. Hence $d_{2n}(t) \geq d_{2n-1}(t)$ for all $n \in N$ and $t > 0$. By (2.11), we have

$$\begin{aligned} &d_{2n}(kt) \\ &\geq a(t)d_{2n-1}(qt) + b(t)d_{2n-1}(qt) + c(t)d_{2n-1}(qt) * d_{2n-1}(qt) \\ &\quad + d(t)d_{2n-1}(qt) + e(t)d_{2n-1}(qt) \\ &= d_{2n-1}(qt) \end{aligned}$$

and so $d_{2n}(kt) \geq d_{2n-1}(qt)$. Thus we have

$$d_n(kt) \geq d_{n-1}(qt), \quad \forall n \geq 1.$$

Therefore, it follows that

$$\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_n, x_n, \frac{q}{k}t) \geq \cdots \geq \mathcal{M}(x_0, x_1, x_1, (\frac{q}{k})^n t).$$

Hence, by Lemma 1.16, $\{x_n\}$ is a Cauchy sequence in X and, by the completeness of X, $\{x_n\}$ converges to a point $x \in X$ and

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} x_{2n+2} = x.$$

Now, we prove that $Sx = x$. In fact, letting $\alpha = 1$, $x = x$ and $y = x_{2n+1}$ in (2.7), respectively, we obtain

$$\begin{aligned} &\mathcal{M}(Sx, Tx_{2n+1}, Tx_{2n+1}, kt) \\ &\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t) \\ &\quad + c(t)\mathcal{M}(x, Tx_{2n+1}, Tx_{2n+1}, t) + d(t)\mathcal{M}(x_{2n+1}, Sx, Sx, t) \\ &\quad + e(t)\mathcal{M}(x, x_{2n+1}, x_{2n+1}, t). \end{aligned} \tag{2.12}$$

If $Sx \neq x$, then, letting $n \longrightarrow \infty$ in (23.12), we have

$$\begin{aligned} &\mathcal{M}(Sx, x, x, kt) \\ &\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x, x, x, t) \\ &\quad + c(t)\mathcal{M}(x, x, x, t) + d(t)\mathcal{M}(x, Sx, Sx, t) + e(t)\mathcal{M}(x, x, x, t) \\ &> \mathcal{M}(x, x, Sx, t), \end{aligned}$$

which is a contradiction. Thus it follows that $Sx = x$.

Similarly, we can prove that $Tx = x$. In fact, again, replacing x by x_{2n} and y by x in (2.7), respectively, for $\alpha = 1$, we have

$$\begin{aligned} &\mathcal{M}(Sx_{2n}, Tx, Tx, kt) \\ &\geq a(t)\mathcal{M}(x_{2n}, Sx_{2n}, Sx_{2n}, t) + b(t)\mathcal{M}(x, Tx, Tx, t) \\ &\quad + c(t)\mathcal{M}(x_{2n}, Tx, Tx, t) + d(t)\mathcal{M}(x, Sx_{2n}, Sx_{2n}, t) \\ &\quad + e(t)\mathcal{M}(x_{2n}, x, x, t) \end{aligned} \tag{2.13}$$

and so, if $Tx \neq x$, letting $n \longrightarrow \infty$ in (2.13), we have

$$\begin{aligned}
&\mathcal{M}(x, Tx, Tx, kt) \\
&\geq a(t)\mathcal{M}(x, x, x, t) + b(t)\mathcal{M}(x, Tx, Tx, t) \\
&\quad + c(t)\mathcal{M}(x, Tx, Tx, t) + d(t)\mathcal{M}(x, x, x, t) + e(t)\mathcal{M}(x, x, x, t) \\
&> \mathcal{M}(x, Tx, Tx, t),
\end{aligned}$$

which implies that $Tx = x$. Therefore, $Sx = Tx = x$ and x is a common fixed point of the self-mappings S and T of X.

The uniqueness of a common fixed point x is easily verified by using the hypothesis. In fact, if x' be another fixed point of S and T, then, for $\alpha = 1$, by (2.7), we have

$$\begin{aligned}
&\mathcal{M}(x, x', x', kt) \\
&= \mathcal{M}(Sx, Tx', Tx', kt) \\
&\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x', Tx', Tx', t) \\
&\quad + c(t)\mathcal{M}(x, Tx', Tx', t) + d(t)\mathcal{M}(x', Sx, Sx, t) + e(t)\mathcal{M}(x, x', x', t) \\
&> \mathcal{M}(x, x', x', t).
\end{aligned}$$

and so $x = x'$. □

Example 2.5. Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space, where $X = [0, 1]$ with t-norm defined $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + |x - y| + |y - z| + |x - z|}, \quad \forall t > 0,\ x, y, z \in X.$$

Define the self-mappings T and S on X as follows:

$$Tx = 1, \quad Sx = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

We can find the functions $a, b, c, d, e : [0, \infty) \longrightarrow [0, 1]$ such that $a(t) + b(t) + c(t) + d(t) + e(t) = 1$ and the following inequality holds:

$$\begin{aligned}
&\mathcal{M}(Sx, Ty, Ty, kt) \\
&\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(y, Ty, Ty, t) \\
&\quad + c(t)\mathcal{M}(x, Ty, Ty, \alpha t) + d(t)\mathcal{M}(y, Sx, Sx, (2 - \alpha)t) \\
&\quad + e(t)\mathcal{M}(x, y, y, t).
\end{aligned}$$

It is easy to see that the all the conditions of Theorem 3.4 hold and 1 is a unique common fixed point of S and T.

From Theorem 2.4, we have the following:

Corollary 2.6. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space with $t*t = t$ for all $t \in [0,1]$. Let S be a mapping from X into itself such that there exists $k \in (0,1)$ such that*

$$\begin{aligned}
&\mathcal{M}(Sx, Sy, Sy, kt)\\
&\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(y, Sy, Sy, t)\\
&\quad + c(t)\mathcal{M}(x, Sy, Sy, \alpha t) + d(t)\mathcal{M}(y, Sx, Sx, (2-\alpha)t)\\
&\quad + e(t)\mathcal{M}(x, y, y, t)
\end{aligned}$$

for all $x, y \in X$ and $\alpha \in (0,2)$, where $a, b, c, d, e : [0,\infty) \longrightarrow [0,1]$ are five functions such that

$$a(t) + b(t) + c(t) + d(t) + e(t) = 1, \quad \forall t \in [0,\infty).$$

Then S have a unique common fixed point in X.

Corollary 2.7. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space with $t*t = t$ for all $t \in [0,1]$. Let S be a mapping from X into itself such that there exists $k \in (0,1)$ such that*

$$\begin{aligned}
&\mathcal{M}(Sx, y, y, kt)\\
&\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x, y, y, \alpha t)\\
&\quad + c(t)\mathcal{M}(y, Sx, Sx, (2-\alpha)t) + d(t)\mathcal{M}(x, y, y, t)
\end{aligned}$$

for all $x, y \in X$ and $\alpha \in (0,2)$, where $a, b, c, d : [0,\infty) \longrightarrow [0,1]$ are five functions such that

$$a(t) + b(t) + c(t) + d(t) = 1, \quad \forall t \in [0,\infty).$$

Then S have a unique common fixed point in X.

Corollary 2.8. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space with $t*t = t$ for all $t \in [0,1]$. Let S and T be mappings from X into itself such that there exists $k \in (0,1)$ such that*

$$\begin{aligned}
&\mathcal{M}(S^n x, T^m y, T^m y, kt)\\
&\geq a(t)\mathcal{M}(x, S^n x, S^n x, t) + b(t)\mathcal{M}(y, T^m y, T^m y, t)\\
&\quad + c(t)\mathcal{M}(x, T^m y, T^m y, \alpha t) + d(t)\mathcal{M}(y, S^n x, S^n x, (2-\alpha)t)\\
&\quad + e(t)\mathcal{M}(x, y, y, t)
\end{aligned}$$

for all $x, y \in X$, $\alpha \in (0,2)$ and $n, m \geq 2$, where $a, b, c, d, e : [0,\infty) \longrightarrow [0,1]$ are five functions such that

$$a(t) + b(t) + c(t) + d(t) + e(t) = 1, \quad \forall t \in [0,\infty).$$

If $S^n T = T S^n$ and $T^m S = S T^m$, then S and T have a unique common fixed point in X.

Proof. By Theorem 2.4, S^n and T^m have a unique common fixed point in X. That is, there exists a unique point $z \in X$ such that $S^n(z) = T^m(z) = z$. Since $S(z) = S(S^n(z)) = S^n(S(z))$ and $S(z) = S(T^m(z)) = T^m(S(z))$, that is, $S(z)$ is fixed point S^n and T^m and so $S(z) = z$. Similarly, $T(z) = z$. This completes the proof. □

Corollary 2.9. *Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space with $t*t = t$ for all $t \in [0,1]$. Let S and T be mappings from X into itself such that there exists $k \in (0,1)$ such that*

$$\mathcal{M}(Sx, Ty, Ty, kt) \geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(y, Ty, Ty, t)$$

for all $x, y \in X$ and $\alpha \in (0,2)$, where $a, b : [0,\infty) \longrightarrow [0,1]$ are two functions such that

$$a(t) + b(t) = 1, \quad \forall t \in [0,\infty).$$

Then S and T have a unique common fixed point in X.

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