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# Observer design for inherently nonlinear systems with lower triangular structure

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***Abstract*— A new observation procedure is proposed for a wide class of single output observable nonlinear systems written in lower triangular form. First, we give the $n$-th order time-varying differentiator that robustly estimates, in asymptotic manner, the higher derivatives of any model-free continuously differentiable signal. This $n$-th order differentiator is a generalization of the time-varying differentiator proposed by the author in [1], [2] and [3]. By using an appropriate change of variables, it is shown that the boundedness of the signal to be differentiated is not necessary for the convergence of the differentiator. Based on the fact that systems written in triangular form are algebraically observable then, the system states can be reproduced through a static diffeomorphism that involves the system input, the system output, and their respective higher derivatives. It is shown that the global convergence of the $n$-th order differentiator implies the asymptotic convergence of the system states without imposing any restrictive condition on the form of nonlinearities.**

***Index Terms*— Nonlinear observer design; Adaptive estimation; Time-varying systems; Signal differentiation.**

## I. INTRODUCTION

STATE estimation of highly nonlinear systems is a longstanding and challenging problem that has been addressed with different looks. The complexity of state reconstruction from the input and the output measurements depends on the system nonlinearities, the nature of the input that may render the system unobservable, and the form of the system output which plays a key role in the stability of the observation error. Until now, there is no unique straightforward method to design an observer for a given nonlinear system. However, under certain conditions, numerous solutions do exist for special forms of systems. By exploiting the structure of the system being observed, the boundedness of the system states or the Lipschitz property of the system nonlinearities, many strategies have been employed to build an observer. Error-linearization-based algorithms [4], [5], [6], [7], Lyapunov design procedures [8], and sliding-mode observer design [9], [10] are among the systematic procedures that have shown satisfactory performances. The reader can also find other challenging procedures as numerical methods [11], neural-network observation techniques [12], algebraic nonlinear observer design [1], and circle-criterion observation methods [13], [14]. When the system fails to be put in certain form of observability, high-gain observer design reveals as a powerful method that is often used to reconstruct the system states under the assumption that the vector nonlinearity is globally or locally Lipschitz, see [15], [16], [17], [18], [19]. However, the Lipschitz constraint is not always verified and prevents generally the global convergence of the high-gain observer. Moreover, the existence of the observer gain is conditioned by the value of the Lipschitz constant which is generally required to be small enough, see [19] for more details. Even though the circle-criterion observer design is conceptually free form the information of the Lipschitz constant [20], [14], this interesting design remains limited to systems with positive-gradient nonlinearities. In this note, a new observation method is given for state estimation of a general class of nonlinear systems satisfying the complete uniform observability condition. The main features of the proposed design are summarized in the following points.

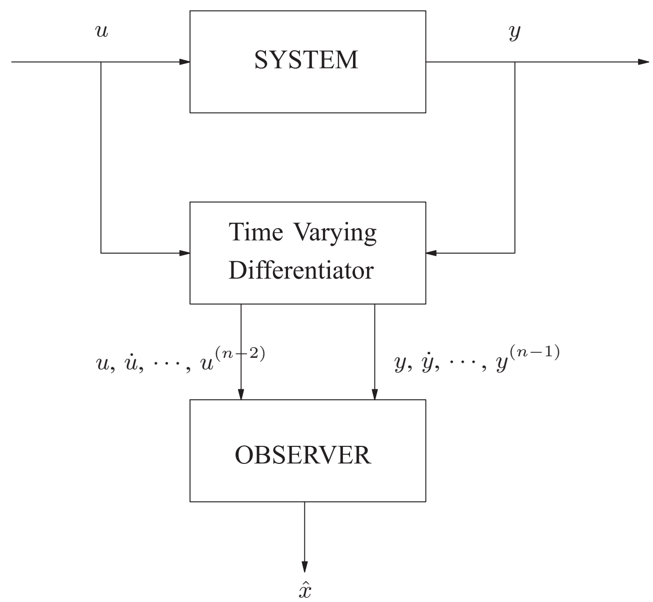


Fig. 1. The scheme of the nonlinear observer

- Robustly estimate the higher derivatives of any differentiable measured signal without incorporating its model or imposing the boundedness of the signal or its respective higher derivatives;
- The proposed $n$-th order differentiator is a generalization of constant-gain differentiators written in controllable canonical form discussed in [2];
- The design procedure is free from any restrictive condition as the Lipschitz or the Hölder conditions generally imposed in high-gain observer design;
- The nonlinearities are not subject to any restrictive condition whenever the uniform observability condition is satisfied;

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- The dynamics of the adaptive algebraic observer is not a copy of the original model with output correction term;
- The convergence of the observation error is global and exponential.

To conceive the dynamics of the whole observer, we start by writing the system states as static algebraic expressions of the input, the output, and their respective higher derivatives. Subsequently, all the variables of the static diffeomorphism, that relate the system unmeasured states to the higher derivatives of the input and the output, are asymptotically estimated. Illustrative example showing the main features of the novel design is discussed. Throughout this paper, we note by $\mathbb{R}$ the set of real numbers. The notation $A > 0$ (resp. $A < 0$) means that the matrix $A$ is positive definite (resp. negative definite). $I_n$ is the identity matrix of appropriate dimension and $A'$ denotes the matrix transpose of $A$. We note by $\triangleq$ any equality by definition. $\dot{x}$ stands for the time-derivative of the vector $x$ with respect to time and $C_n^k$ stands for the binomial coefficient.

## II. System description and observability analysis

Consider the class of dynamical systems written in the lower-triangular form:

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1, u), \\ &\vdots \\ \dot{x}_i &= x_{i+1} + f_i(x_1, x_2, \cdots, x_i, u), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \cdots, x_n, u), \\ y &= x_1, \end{aligned} \tag{1}$$

where $u \in \mathbb{R}^m$ is the system input and $y \in \mathbb{R}$ is the system output. In order to complete the description of the considered system, let us begin by considering the following assumptions.

*Assumption 1:* Each nonlinearity $f_i(x_1, x_2, \cdots, x_i, u)$, $1 \leq i \leq n$ is a well-defined nonlinearity with respect to their variables $x_1$, $x_2$, $\cdots$, $x_i$, $u$.

*Assumption 2:* For a given bounded input $u \in \mathbb{R}^m$, the system states do not leave any compact set. In other words, the system trajectories are well-defined for all $t \geq 0$ such that for any instant $t \geq 0$, we can find a large compact set $\Omega_t$ where the system states live in.

Before starting the analysis of the system observability along with the observer design, let us introduce the following definitions.

*Definition 1:* Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x, u), \\ y = h(x), \end{cases} \tag{2}$$

where $x = x(t) \in \mathscr{M} \subset \mathbb{R}^n$ represents the system state vector, $f(\cdot, \cdot)$ is smooth vector with $f(0, 0) = 0$ and $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ is the control input. The output nonlinearity $y = y(t) = h(x(t)) \in \mathbb{R}^p$ is supposed to be smooth with $h(0) = 0$. We say that system (2) is observable if for every two different initial conditions $x_0$ and $\bar{x}_0$ there exist an interval $[0,\, T]$, $T \in \mathbb{R}_{>0}$ and an admissible control $u(t)$ defined on $[0,\, T]$ such that the associated outputs $y(x_0, u(t))$, $y(\bar{x}_0, u(t))$ are not identically equal on $[0,\, T]$. We say, in this case, that the control input $u(t)$ distinguishes the pair $(x_0, \bar{x}_0)$ on $[0,\, T]$.

*Definition 2:* Consider system (2). The control input $u(t) \in \mathscr{U} \subset \mathbb{R}^m$ is said universal on $[0,\, T]$, if it distinguishes every different initial states $(x_0, \bar{x}_0)$ on $[0,\, T]$.

*Definition 3:* System (2) is said uniformly observable if every admissible control $u(t)$ defined on $[0,\, T]$, is a universal one.

*Definition 4:* System (2) is said to be algebraically observable if there exist two positive integers $\mu$ and $\nu$ such that

$$x(t) = \phi\left(y,\, \dot{y},\, \ddot{y}, \cdots,\, y^{(\mu)},\, u,\, \dot{u},\, \ddot{u}, \cdots,\, u^{(\nu)}\right)(t), \tag{3}$$

where $\phi(\cdot) : \mathbb{R}^{(\mu+1)p} \times \mathbb{R}^{(\nu+1)m} \mapsto \mathbb{R}^n$ is a differentiable vector valued nonlinearity that depends on the inputs, the outputs, and their respective higher derivatives.

According to the above definitions, system (1) is uniformly observable for any input. This property can be also checked via the algebraic observability of the system since the state vector is reconstructed by recurrence as follows:

$$\begin{aligned} x_1 &= y = \varphi_1(y), \\ x_2 &= \dot{y} - f_1(y, u) = \varphi_2(y, \dot{y}, u), \\ &\vdots \\ x_i &= \dot{\varphi}_{i-1}(y, \dot{y}, \cdots, y^{(i-2)}, u, \dot{u}, \cdots, u^{(i-3)}) \\ &\quad - f_{i-1}\big(y, \varphi_2(y, \dot{y}, u), \cdots, \varphi_{i-1}(y, \dot{y}, \cdots, y^{(i-2)}, \\ &\qquad\qquad u, \dot{u}, \cdots, u^{(i-3)}), u\big), \\ &= \varphi_i(y, \dot{y}, \cdots, y^{(i-1)}, u, \dot{u}, \cdots, u^{(i-2)}), \\ &\vdots \\ x_n &= \dot{\varphi}_{n-1}(y, \dot{y}, \cdots, y^{(n-2)}, u, \dot{u}, \cdots, u^{(n-3)}) \\ &\quad - f_{n-1}\big(y, \varphi_2(y, \dot{y}, u), \cdots, \varphi_{n-1}(y, \dot{y}, \cdots, y^{(n-2)}, \\ &\qquad\qquad u, \dot{u}, \cdots, u^{(n-3)}), u\big), \\ &= \varphi_n\big(y, \dot{y}, \cdots, y^{(n-1)}, u, \dot{u}, \cdots, u^{(n-2)}\big). \end{aligned} \tag{4}$$

Since all the nonlinearities $f_i(x_1, \cdots, x_i)$, $1 \leq i \leq n$ are continuously differentiable then, the resulting functions $\varphi_i(s)$, $1 \leq i \leq n$ are well-defined with respect to their variables; hence, the states can be reconstructed from the information of the inputs and output without any singularity.

## III. Observer analysis

### A. n-th order time-varying differentiator

In this section, we present an adaptive-like differentiator whose successive states converge asymptotically to the successive higher derivatives of the input signal $y$. We show that the convergence of the differentiator is always assured even when $y$ is not bounded. Before presenting this result let us introduce the following Lemma.

*Lemma 1:* Let $P(\gamma)$ and $R(\gamma)$ be two parameter-dependent matrices verifying the following coupled Lyapunov-like equations:

$$\begin{aligned}&-\gamma P(\gamma)-P(\gamma)A'-AP(\gamma)+BB'=0,\\ &\gamma R(\gamma)+(A-BB'P^{-1}(\gamma))'R(\gamma)\\ &\qquad\qquad+R(\gamma)(A-BB'P^{-1}(\gamma))+C'C=0,\end{aligned} \tag{5}$$

where $A$, $B$ and $C$ are defined as

$$A\triangleq\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&0\\0&0&0&\cdots&1\\0&0&0&\cdots&0\end{bmatrix},B\triangleq\begin{bmatrix}0\\0\\0\\ \vdots\\1\end{bmatrix},C\triangleq\begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix}'. \tag{6}$$

Then, for $\gamma>0$, we have

- $P(\gamma)$ and $R(\gamma)$ are positive definite and are given by:
$$P(\gamma)=\frac{1}{\gamma}G(\gamma)P_1G(\gamma),\ R(\gamma)=\frac{1}{\gamma}S(\gamma)R_1S(\gamma) \tag{7}$$
where the matrices $P_1$, $R_1$, $G(\gamma)$, $S(\gamma)$ are defined as $P_1\triangleq P(\gamma)|_{\gamma=1}$, $R_1\triangleq R(\gamma)|_{\gamma=1}$, $G(\gamma)\triangleq\mathrm{diag}\Big(\frac{1}{\gamma^{n-1}},\frac{1}{\gamma^{n-2}},\cdots,\frac{1}{\gamma},1\Big)$, and $S(\gamma)=\mathrm{diag}\Big(1,\frac{1}{\gamma},\cdots,\frac{1}{\gamma^{n-1}}\Big)$.
- The matrices $\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)$ and $\frac{\mathrm{d}}{\mathrm{d}\gamma}R(\gamma)$ are negative definite.
- The matrices $\frac{\mathrm{d}}{\mathrm{d}\gamma}P^{-1}(\gamma)$ and $\frac{\mathrm{d}}{\mathrm{d}\gamma}R^{-1}(\gamma)$ are positive definite.

*Proof:* The matrix $P(\gamma)$ is symmetric and positive definite for all $\gamma>0$ because the matrix $\Upsilon(\gamma)=-\frac{\gamma}{2}I-A$ is Hurwitz and verifies the following Lyapunov matrix equation

$$\Upsilon(\gamma)\,P(\gamma)+P(\gamma)\,\Upsilon'(\gamma)=-BB'. \tag{8}$$

Since all the eigenvalues of the matrix $\Xi(\gamma)=\frac{\gamma}{2}I+A-BB'P^{-1}(\gamma)$ are equal to $-\frac{\gamma}{2}$, and the matrix $\Xi(\gamma)$ verifies the Lapunov equation

$$\Xi'(\gamma)R(\gamma)+R(\gamma)\Xi(\gamma)=-C'C \tag{9}$$

then, the matrix $R(\gamma)$ is symmetric and positive-definite for all $\gamma>0$. The matrix $P_1$ verifies the following Lypunov-like equation:

$$-P_1-P_1A'-AP_1+BB'=0. \tag{10}$$

Pre- and post multiplying the last matrix equation by $G(\gamma)$, we obtain

$$\begin{aligned}&-G(\gamma)P_1G(\gamma)-G(\gamma)P_1A'G(\gamma)-G(\gamma)AP_1\\ &\qquad\qquad+G(\gamma)BB'G(\gamma)=0.\end{aligned} \tag{11}$$

Using the following properties $\gamma G(\gamma)A=AG(\gamma)$, $\gamma A'G(\gamma)=G(\gamma)A'$, $B'G(\gamma)=B'$, $G(\gamma)B=B$, we have

$$\begin{aligned}&-\gamma\Big[\frac{1}{\gamma}G(\gamma)P_1G(\gamma)\Big]-\Big[\frac{1}{\gamma}G(\gamma)P_1G(\gamma)\Big]A'\\ &\qquad-A\Big[\frac{1}{\gamma}G(\gamma)P_1G(\gamma)\Big]+BB'=0.\end{aligned} \tag{12}$$

By comparison of the last matrix equation with first matrix in (5), we conclude that $P(\gamma)=\frac{1}{\gamma}G(\gamma)P_1G(\gamma)$.

Similarly, the matrix $R_1$ is the solution of the following equation:

$$R_1+(A'-P_1^{-1}BB')R_1+R_1(A-BB'P_1^{-1})+C'C=0. \tag{13}$$

Pre- and post multiplying Eq. (13) by $S(\gamma)$, we obtain

$$\begin{aligned}&S(\gamma)R_1S(\gamma)+S(\gamma)(A'-P_1^{-1}BB')R_1S(\gamma)\\ &+S(\gamma)R_1(A-BB'P_1^{-1})S(\gamma)+S(\gamma)C'CS(\gamma)=0.\end{aligned} \tag{14}$$

Using the following properties: $S(\gamma)C'=C'$, $S(\gamma)A'=\frac{1}{\gamma}A'S(\gamma)$, $AS(\gamma)=\frac{1}{\gamma}S(\gamma)A$, Eq. (14) takes the form:

$$\begin{aligned}&\gamma\Big[\frac{1}{\gamma}S(\gamma)R_1S(\gamma)\Big]+\Big(\frac{1}{\gamma}A'S(\gamma)-S(\gamma)P_1^{-1}BB'\Big)R_1S(\gamma)\\ &+S(\gamma)R_1\Big(\frac{1}{\gamma}S(\gamma)A-BB'P_1^{-1}S(\gamma)\Big)+C'C=0.\end{aligned} \tag{15}$$

Using the fact that $S(\gamma)B=\frac{1}{\gamma^{n-1}}B$, $B'G^{-1}(\gamma)=B'$ and $G^{-1}(\gamma)=\gamma^{n-1}S(\gamma)$ then, we can write that

$$\begin{aligned}BB'P_1^{-1}S(\gamma)&=\frac{1}{\gamma}S(\gamma)BB'\Big(\gamma G^{-1}(\gamma)P_1^{-1}G^{-1}(\gamma)\Big)\\ &\frac{1}{\gamma}S(\gamma)BB'P^{-1}(\gamma).\end{aligned} \tag{16}$$

Based on the last relation, Eq. (15) becomes

$$\begin{aligned}&\gamma\Big[\frac{1}{\gamma}S(\gamma)R_1S(\gamma)\Big]+\Big(A-BB'P^{-1}(\gamma)\Big)'\Big[\frac{1}{\gamma}S(\gamma)R_1S(\gamma)\Big]\\ &+\Big[\frac{1}{\gamma}S(\gamma)R_1S(\gamma)\Big]\Big(A-BB'P^{-1}(\gamma)\Big)+C'C=0.\end{aligned} \tag{17}$$

By comparison of the last equation with the second equation in (5), we conclude immediately that $R(\gamma)=\frac{1}{\gamma}S(\gamma)R_1S(\gamma)$. This ends the proof of item $i$) of the Lemma.

$ii$) By doing the differentiation of $P(\gamma)$ in the following manner

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)&=\Big[\frac{1}{\gamma}G(\gamma)\Big]P_1\Big[\frac{1}{\gamma}G(\gamma)\Big]\\ &+\gamma\frac{\mathrm{d}}{\mathrm{d}\gamma}\Big[\frac{1}{\gamma}G(\gamma)\Big]P_1\Big[\frac{1}{\gamma}G(\gamma)\Big]\\ &+\gamma\Big[\frac{1}{\gamma}G(\gamma)\Big]P_1\frac{\mathrm{d}}{\mathrm{d}\gamma}\Big[\frac{1}{\gamma}G(\gamma)\Big]\end{aligned} \tag{18}$$

and taking into account that

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}\left[\frac{1}{\gamma}G(\gamma)\right] = -\frac{1}{\gamma^2}\Gamma_g\, G(\gamma),\, \Gamma_g = \mathrm{diag}(n, n-1, \cdots, 1). \tag{19}$$

By substitution of (19) into (18), we have

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) = -\left[\frac{1}{\gamma}G(\gamma)\right]\left[\left(\Gamma_g - \frac{1}{2}I\right)P_1 + \left(\Gamma_g - \frac{1}{2}I\right)P_1\right] \times \left[\frac{1}{\gamma}G(\gamma)\right]. \tag{20}$$

Since the matrix $\left(\Gamma_g - \frac{1}{2}I\right)P_1 + \left(\Gamma_g - \frac{1}{2}I\right)P_1 > 0$, then $\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma) < 0$, $\forall\gamma > 0$.

Actually, $R(\gamma) = \gamma\left[\frac{1}{\gamma}S(\gamma)\right]R_1\left[\frac{1}{\gamma}S(\gamma)\right]$, and $\frac{\mathrm{d}}{\mathrm{d}\gamma}\left[\frac{1}{\gamma}S(\gamma)\right] = -\frac{1}{\gamma^2}\Gamma_s S(\gamma)$ where $\Gamma_s \triangleq \left(1, 2, \cdots, n\right)$. Then, one can easily show that

$$\frac{\mathrm{d}}{\mathrm{d}\gamma}R(\gamma) = -\left[\frac{1}{\gamma}S(\gamma)\right]\left[\left(\Gamma_s - \frac{1}{2}I\right)R_1 + \left(\Gamma_s - \frac{1}{2}I\right)R_1\right] \times \left[\frac{1}{\gamma}S(\gamma)\right], \tag{21}$$

which is negative definite due to the fact that $\left(\Gamma_s - \frac{1}{2}I\right)R_1 + \left(\Gamma_s - \frac{1}{2}I\right)R_1 > 0$.

*iii)* Since $\forall\gamma > 0$, the derivative matrices are given by $\frac{\mathrm{d}}{\mathrm{d}\gamma}P^{-1}(\gamma) = -P^{-1}(\gamma)\frac{\mathrm{d}}{\mathrm{d}\gamma}P(\gamma)P^{-1}(\gamma)$, $\frac{\mathrm{d}}{\mathrm{d}\gamma}R^{-1}(\gamma) = -R^{-1}(\gamma)\frac{\mathrm{d}}{\mathrm{d}\gamma}R(\gamma)R^{-1}(\gamma)$ and based on the result of the last item *ii)*, we conclude that the inverse of the derivative matrices are positive definite. This ends the proof of Lemma 1.

The design of the time-varying differentiator is summarized in the following statement.

*Theorem 1:* Let $y$ be a continuous function having $n$ continuous well-defined derivatives noted: $\dot{y}$, $\ddot{y}$, $\cdots$, $y^{(n)}$. Let $\hat{y} = \arctan(y)$ and let $\phi_1(s_1)$, $\phi_2(s_1, s_2)$, $\cdots$, $\phi_n(s_1, s_2, \cdots, s_n)$ be $n$ scalar functionals defined as

$$\begin{aligned} &y = \phi_1(\hat{y}),\, \dot{y} = \phi_2(y, \dot{\hat{y}}),\, \ddot{y} = \phi_3(y, \dot{\hat{y}}, \ddot{\hat{y}}), \cdots, \\ &y^{(n)} = \phi_n(y, \dot{\hat{y}}, \ddot{\hat{y}}, \cdots, \hat{y}^{(n)}). \end{aligned} \tag{22}$$

Let $\xi_1, \xi_2, \cdots, \xi_n$ be the states of the following time-varying system:

$$\begin{aligned} &\dot{\xi}_1 = \xi_2, \cdots, \dot{\xi}_i = \xi_{i+1}, \cdots, \\ &\dot{\xi}_n = -C_n^1\gamma^n(\xi_1 - \arctan(y)) - \sum_{i=1}^{n-1} C_n^{n-i}\gamma^{n-i}\xi_{i+1}, \\ &\dot{\gamma} = \begin{cases} \alpha, & \text{if, } |\xi_1 - \arctan(y)| \neq 0,\, \alpha > 1,\, \gamma(0) > 0, \\ 0, & \text{if, } |\xi_1 - \arctan(y)| = 0, \end{cases} \end{aligned} \tag{23}$$

then,

$$\lim_{t\to\infty} y^{(i)} - \phi_i(y, \xi_2, \cdots, \xi_i) = 0,\;\; 2 \leq i \leq n. \tag{24}$$

*Proof:* To prove this result it is sufficient to prove that

$$\lim_{t\to\infty} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\arctan(y)\right) = \xi_{k+1},\; 0 \leq k \leq n. \tag{25}$$

To obtain the functionals $(\phi_i)_{2\leq i\leq n}$ that link the derivatives of $\hat{y}$ to the derivatives of $y$, one can easily generate by differentiation of $\hat{y}$ and substitution the following first functions:

$$\begin{aligned} &\hat{y} = \arctan(y), \\ &\dot{y} = (1 + y^2)\dot{\hat{y}} = \phi_2(y, \dot{\hat{y}}), \\ &\ddot{y} = 2y(1 + y^2)\dot{\hat{y}}^2 + (1 + y^2)\ddot{\hat{y}} = \phi_3(y, \dot{\hat{y}}, \ddot{\hat{y}}). \end{aligned} \tag{26}$$

The model-free signal $\hat{y}$ can be seen as the output of the following system:

$$\begin{aligned} \dot{z} &= A\, z + B\hat{y}^{(n)}, \\ \hat{y} &= C\, z, \end{aligned} \tag{27}$$

where $z \in \mathbb{R}^n$. In matrix form, the time-varying differentiator can be written

$$\begin{aligned} &\dot{\xi} = A\,\xi - BB'P^{-1}(\gamma)\left(\xi - C'\arctan(y)\right), \\ &0 = -\gamma P(\gamma) - P(\gamma)A' - AP(\gamma) + BB', \\ &\dot{\gamma} = \begin{cases} \alpha, & \text{if, } |\xi_1 - \arctan(y)| \neq 0,\, \alpha > 1,\, \gamma(0) > 0, \\ 0, & \text{if, } |\xi_1 - \arctan(y)| = 0, \end{cases} \end{aligned} \tag{28}$$

By writing the dynamics of the differentiator as

$$\dot{\xi} = A\,\xi - BB'P^{-1}(\gamma)(\xi - z) + BB'P^{-1}(\gamma)(C'\hat{y} - z). \tag{29}$$

Let $e_1$, $e_2$, $\cdots$, $e_n$ be the canonical base column vectors of dimension $n$. Then, the difference $C'\hat{y} - z$ can be rewritten as

$$C'\hat{y} - z = -\sum_{i=1}^{n-1} e_{i+1}\hat{y}^{(i)}. \tag{30}$$

Define $\tilde{x} = \xi - z$, then based on (27), (29), (30), we have

$$\dot{\tilde{x}} = (A - BB'P^{-1}(\gamma))\tilde{x} - BB'P^{-1}(\gamma)\sum_{i=1}^{n-1} e_{i+1}\hat{y}^{(i)} - e_n\hat{y}^{(n)}. \tag{31}$$

Let us associate the Lyapunov function $V(\tilde{x}) = \tilde{x}'R(\gamma)\tilde{x}$ where $R(\gamma)$ is defined as in Eqs. (5). Then,

$$\dot{V}(\tilde{x}) = \dot{\tilde{x}}'R(\gamma)\tilde{x} + \tilde{x}'R(\gamma)\dot{\tilde{x}}' + \tilde{x}'\left[\dot{\gamma}\frac{\mathrm{d}}{\mathrm{d}\gamma}R(\gamma)\right]\tilde{x} \tag{32}$$

Using the result of Lemma 1, the term $\tilde{x}'\left[\dot{\gamma}\frac{\mathrm{d}}{\mathrm{d}\gamma}R(\gamma)\right]\tilde{x} < 0$ for $\tilde{x} \neq 0$. Therefore,

$$\begin{aligned}\dot{V}(\tilde{x}) &\leq \dot{\tilde{x}}'R(\gamma)\tilde{x} + \tilde{x}'R(\gamma)\dot{\tilde{x}}' \\ &\leq \tilde{x}'(A - BB'P^{-1}(\gamma))'R(\gamma)\tilde{x} \\ &+ \tilde{x}'R(\gamma)(A - BB'P^{-1}(\gamma)) \\ &- 2\,\tilde{x}'R(\gamma)BB'P^{-1}(\gamma)\sum_{i=1}^{n-1} e_{i+1}\hat{y}^{(i)} - 2\tilde{x}'R(\gamma)e_n\hat{y}^{(n)}.\end{aligned} \tag{33}$$

Using (5), we have

$$\begin{aligned}\dot{V}(\tilde{x}) &\leq \tilde{x}'(-\gamma R(\gamma) - BB')\tilde{x} \\ &- 2\,\tilde{x}'R(\gamma)BB'P^{-1}(\gamma)\sum_{i=1}^{n-1} e_{i+1}\hat{y}^{(i)} - 2\tilde{x}'R(\gamma)e_n\hat{y}^{(n)}.\end{aligned} \tag{34}$$

Since the Cholesky decomposition of the matrix $R(\gamma) = \tilde{R}(\gamma)\tilde{R}'(\gamma)$ where $\tilde{R}(\gamma) = \dfrac{S(\gamma)}{\sqrt{\gamma}}R_1^{\frac{1}{2}}$ where $R_1^{\frac{1}{2}}$ is the square root of the matrix $R_1$. Then, using the result of Lemma 1, the derivative of the Lyapunov function is bounded as follows

$$\begin{aligned}&\dot{V}(\tilde{x}) \leq -\gamma V(\tilde{x}) \\ &- 2\sum_{i=1}^{n-1}\tilde{x}'S(\gamma)R_1S(\gamma)BB'G^{-1}(\gamma)P_1^{-1}G^{-1}(\gamma)e_{i+1}\hat{y}^{(i)} \\ &- \frac{2}{\gamma}\tilde{x}'S(\gamma)R_1S(\gamma)\,e_n\,\hat{y}^{(n)} \\ &\leq -\gamma V(\tilde{x}) + 2\left\|\tilde{x}'\frac{S(\gamma)}{\sqrt{\gamma}}R_1^{\frac{1}{2}}\right\| \\ &\times\left\|\sum_{i=1}^{n-1}R_1^{\frac{1}{2}}\frac{S(\gamma)}{\sqrt{\gamma}}BB'G^{-1}(\gamma)P_1^{-1}G^{-1}(\gamma)\,e_{i+1}\hat{y}^{(i)}\right\| \\ &+ \frac{2}{\sqrt{\gamma}}\left\|\tilde{x}'\frac{S(\gamma)}{\sqrt{\gamma}}R_1^{\frac{1}{2}}\right\|\left\|R_1^{\frac{1}{2}}S(\gamma)\,e_n\,\hat{y}^{(n)}\right\|\end{aligned} \tag{35}$$

This gives

$$\begin{aligned}&\dot{V}(\tilde{x}) \leq -\gamma V(\tilde{x}) + 2\sqrt{V(\tilde{x})}\times \\ &\times\left\|\sum_{i=1}^{n-1}R_1^{\frac{1}{2}}\frac{S(\gamma)}{\sqrt{\gamma}}BB'G^{-1}(\gamma)P_1^{-1}G^{-1}(\gamma)\,e_{i+1}\hat{y}^{(i)}\right\| \\ &+ \frac{2}{\sqrt{\gamma}}\sqrt{V(\tilde{x})}\left\|R_1^{\frac{1}{2}}S(\gamma)\,e_n\,\hat{y}^{(n)}\right\|\end{aligned} \tag{36}$$

Using the relations: $S(\gamma)B = \dfrac{1}{\gamma^{n-1}}B$, $G^{-1}(\gamma) = \gamma^{n-1}S(\gamma)$ and $B'G^{-1}(\gamma) = B'$ then

$$\begin{aligned}&\left\|\sum_{i=1}^{n-1}R_1^{\frac{1}{2}}\frac{S(\gamma)}{\sqrt{\gamma}}BB'G^{-1}(\gamma)P_1^{-1}G^{-1}(\gamma)\,e_{i+1}\hat{y}^{(i)}\right\| \\ &= \left\|\sum_{i=1}^{n-1}\frac{R_1^{\frac{1}{2}}}{\sqrt{\gamma}}BB'P_1^{-1}S(\gamma)\,e_{i+1}\hat{y}^{(i)}\right\|\end{aligned} \tag{37}$$

Since $\hat{y}$ and its higher derivatives are bounded whatever $y$ then $\forall\ \gamma > 1$, we can always find two constants $c_1$ and $c_2$ such that

$$\begin{aligned}&\left\|\sum_{i=1}^{n-1}\frac{R_1^{\frac{1}{2}}}{\sqrt{\gamma}}BB'P_1^{-1}S(\gamma)\,e_{i+1}\hat{y}^{(i)}\right\| \leq \frac{c_1}{\gamma^{\frac{3}{2}}}, \\ &\left\|R_1^{\frac{1}{2}}S(\gamma)\,e_n\,\hat{y}^{(n)}\right\| \leq \frac{c_2}{\gamma^n}.\end{aligned} \tag{38}$$

This implies that

$$\dot{V}(\tilde{x}) \leq -\gamma V(\tilde{x}) + \left(\frac{2\,c_1}{\gamma^{\frac{3}{2}}} + \frac{2c_2}{\gamma^{n+\frac{1}{2}}}\right)\sqrt{V(\tilde{x})} \tag{39}$$

Since for $\xi_1 - \hat{y} \neq 0$, and $t \geq 0$ we have $\gamma(t) > \alpha\,t$ then, for $\xi_1 - \hat{y} \neq 0$, the derivative of the Lyapunov function can be bounded as follows

$$\dot{V}(\tilde{x}) \leq -\alpha\,t\,V(\tilde{x}) + \left(\frac{2\,c_1}{t^{\frac{3}{2}}} + \frac{2c_2}{t^{n+\frac{1}{2}}}\right)\sqrt{V(\tilde{x})} \tag{40}$$

Let $W(\tilde{x}) = \sqrt{V(\tilde{x})}$. This gives

$$\dot{W}(\tilde{x}) \leq -\frac{\alpha}{2}\,t\,W(\tilde{x}) + \left(\frac{c_1}{t^{\frac{3}{2}}} + \frac{c_2}{t^{n+\frac{1}{2}}}\right). \tag{41}$$

This implies that

$$\begin{aligned}&\int_0^t\left[\dot{W}(\tilde{x}(t))\,e^{\frac{\alpha}{4}t^2} + \frac{\alpha}{2}W(\tilde{x}(t))\,e^{\frac{\alpha}{4}t^2}\right]\mathrm{d}t \leq \\ &\qquad\int_0^t\left(\frac{c_1}{t^{\frac{3}{2}}} + \frac{c_2}{t^{n+\frac{1}{2}}}\right)e^{\frac{\alpha}{4}t^2}\mathrm{d}t.\end{aligned} \tag{42}$$

Consequently,

$$W(\tilde{x}) \leq W(0)\,e^{-\frac{\alpha}{4}t^2} + e^{-\frac{\alpha}{4}t^2}\int_0^t\left(\frac{c_1}{t^{\frac{3}{2}}} + \frac{c_2}{t^{n+\frac{1}{2}}}\right)e^{\frac{\alpha}{4}t^2}\mathrm{d}t. \tag{43}$$

Since $\lim_{t\to\infty} e^{-\frac{\alpha}{4}t^2}\int_0^t\left(\frac{c_1}{t^{\frac{3}{2}}} + \frac{c_2}{t^{n+\frac{1}{2}}}\right)e^{\frac{\alpha}{4}t^2}\mathrm{d}t = 0$ then, we conclude that whatever $\xi_1 - \hat{y} \neq 0$, $W(\tilde{x})$ becomes after a transient short period strictly decreasing until the condition $\xi_1 - \hat{y} = 0$ is verified. From the dynamics of the differentiator (1), the condition $\xi_1 - \hat{y} = 0$ implies that $\xi_i = \hat{y}^{(i-1)}$, $2 \leq i \leq n$. This ends the proof of the Theorem.

### B. The observer

Based on the previous results, we summarize the design of the nonlinear observer in the following statement.

*Theorem 2:* Consider system (1) under Assumptions 1-2. Let $\big(\varphi_i(\cdot)\big)_{1\leq i\leq n}$ and $\big(\phi_i(\cdot)\big)_{1\leq i\leq n-1}$ be smooth functionals such that all the system states are given by:

$$x_i = \varphi_i(y, \dot{y}, \cdots, y^{(i-1)}, u, \dot{u}, \cdots, u^{(i-2)}),\ 2 \leq i \leq n, \tag{44}$$

and

$$\left\{\begin{array}{l}\hat{y} = \arctan(y), \\ y^{(i-1)} = \phi_i(y, \dot{\hat{y}}, \ddot{\hat{y}}, \cdots, \hat{y}^{(i-1)}),\ 1 \leq i \leq n.\end{array}\right. \tag{45}$$

Define the nonlinear observer as

$$\begin{cases} \dot{\xi} = A\xi - BB'P^{-1}(\gamma)\Big(\xi - C'\arctan(y)\Big), \\ 0 = -\gamma P(\gamma) - P(\gamma)A' - AP(\gamma) + BB', \\ \dot{\gamma} = \begin{cases} \alpha, & \text{if, } |\xi_1 - \arctan(y)| \neq 0,\ \alpha > 1,\ \gamma(0) > 0, \\ 0, & \text{if, } |\xi_1 - \arctan(y)| = 0, \end{cases} \\ \hat{x}_1 = y, \\ \hat{x}_i = \varphi_i(y, \phi_2(y,\xi_2), \phi_3(y,\xi_2,\xi_3), \cdots, \\ \qquad \cdots, \phi_i(y,\xi_2,\xi_3,\cdots,\xi_i), u, \dot{u}, \cdots, u^{(i-2)}) \end{cases} \tag{46}$$

where $A$, $B$ and $C$ are defined as in (6). Then,

$$\lim_{t\to\infty}(x_i - \hat{x}_i) = 0;\ 1 \leq i \leq n. \tag{47}$$

*Proof:* Based on the results of Lemma 1 and Theorem 1, the state vector $\xi$ converges asymptotically to the successive higher derivatives of the bounded input $\hat{y} = \arctan(y)$, i.e., $\lim_{t\to\infty} \xi_i - \hat{y}^{(i-1)} = 0,\ 1 \leq i \leq n$. Since the observer states $(\hat{x}_i)_{2\leq i\leq n}$ given in (46) are algebraic expressions that involve the $\xi$-states then, the convergence of the time-varying differentiator implies the convergence of the estimated states to the true ones. For the particular case where $n = 2$, the observer is reduced to the LTV differentiator discussed in [1]. The proposed differentiation scheme is robust against measurement error since the output injection is bounded and present just in the last equation of the differentiator dynamics.

## IV. Example

Consider the nonlinear system

$$\dot{x}_1 = x_2, \dot{x}_2 = x_3, \dot{x}_3 = -x_3^3 - x_1^3 x_2 + u, y = x_1. \tag{48}$$

From the dynamical equations (48), we have $x_1 = y = \varphi_1(y)$, $x_2 = \dot{y} = \varphi_2(\dot{y})$, $x_3 = \ddot{y} = \varphi_3(\ddot{y})$. Using the following relations $y = \phi_1(y)$, $\dot{y} = (1+y^2)\dot{\hat{y}} = \phi_2(y,\dot{\hat{y}})$, $\ddot{y} = 2y(1+y^2)\dot{\hat{y}}^2 + (1+y^2)\ddot{\hat{y}} = \phi_3(y,\dot{\hat{y}},\ddot{\hat{y}})$. Then, $x_1 = y = \phi_1(y)$, $x_2 = \phi_2(y,\dot{\hat{y}})$, $x_3 = \phi_3(y,\dot{\hat{y}},\ddot{\hat{y}})$. Based on the last relation, the adaptive algebraic observer is readily constructed as

$$\begin{aligned} &\dot{\xi}_1 = \xi_2,\ \dot{\xi}_2 = \xi_3, \\ &\dot{\xi}_3 = -\gamma^3\big(\xi_1 - \arctan(y)\big) - 3\gamma^2\xi_2 - 3\gamma\xi_3, \\ &\dot{\gamma} = \begin{cases} \alpha & \text{if, } |\arctan(y) - \xi_1| \neq 0,\ \gamma(0) \geq 1, \alpha \geq 1, \\ 0 & \text{otherwise.} \end{cases} \\ &\hat{x}_1 = y,\ \hat{x}_2 = \phi_2(y,\xi_2),\ \hat{x}_3 = \phi_3(y,\xi_2,\xi_3). \end{aligned} \tag{49}$$

## V. Conclusion

In this paper, a new nonlinear observer is proposed for uniformly observable systems given in lower triangular form. It is showed that the existence of the observer is not realted to conservative conditions as the restriction of the form of the nonlinearities, the boundedness of the system states or the Lipschitz condition that is generally imposed in high-gain observer design. A new robust differentiation scheme is given to reproduce both the higher derivatives of the system outputs and the unmeasured states that are related algebraically to the system input and the output derivatives. The proposed $n$-th order differentiation scheme generalizes the result given in [1] without concatenation of several differentiators which allows the reduction of the observer states by $n$.

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