2 The Mean-Variance Optimization Model

In the first section of this chapter, we will briefly summarize the economic principles that form the basis for mean-variance portfolio selection. An introduction similar to the one presented here can be found in any of the standard references for financial economics (see e.g. Huang and Litzenberger [HL88], or LeRoy and Werner [LW01]) or in the books from Markowitz [Mar59, Mar87].
Section 2.2 introduces the mean-variance model for portfolio selection. It consists of two optimization criteria (variance and expected return) and an arbitrary number of linear equations and inequalities. Three different approaches how to compute a solution for this model are presented: the ϵ-Constraint approach, the weighting method, and parametric quadratic programming. In Section 2.3 we describe other possible dispersion measures besides the variance which better capture the notion of risk, and Section 2.4 explains the origin of the benchmark problems we will use for testing in the remainder of the thesis. In Section 2.5, which concludes this chapter, we specify several types of constraints that may be relevant for portfolio selection problems, we categorize them and analyze their influence on the difficulty of the optimization process.

2.1 Fundamentals of Portfolio Selection

In a market economy, nearly everybody regularly has to solve a variation of the problem that lies at the core of portfolio selection: what to do with a given amount of money in order to get the highest degree of overall well-being. This problem description is very vague. In order to handle it quantitatively, several additional assumptions, simplifications, and formalizations have to be made.
In economics, "well-being" is often measured with the help of a *utility function* $u : Y \mapsto \mathbb{R}$, that maps every possible outcome Y for an event to a real number. A higher objective function value indicates a higher degree of well-being.
The first assumption we make – which is rather general – is that the investor is only interested in financial gain. Other motivations, like e.g. the preference of investments that are ethically unobjectionable, are not considered.
Another important simplification is the assumption that the investment process can be expressed as a so called *one-period model*. In a one-period model, the investment decision is taken at a point in time t_0, and during the period $\triangle t$ the decision is not or can not be revised. At $t_1 = t_0 + \triangle t$, each investment offers a specific yield. The investor's goal in this model is to maximize his end-of-period wealth W_1. What makes this decision problem nontrivial is that for some or all investments the end-of-period yield is not known in t_0, which makes the problem non-deterministic.
One-period models certainly have serious drawbacks, as it is hardly imaginable that an investor will stand by and do nothing if she receives important information during the

period $\triangle t$ that would cause her to adapt her investment positions to the new circumstances. Unfortunately, more advanced models that allow multiperiod transactions or even continuous buying and selling introduce a degree of complexity that is not easy to handle. They also require either additional information, e.g. about the consumption preferences of the investor, or they make general assumptions in that direction (e.g. only the terminal wealth is of interest). As the algorithms allowing the integration of complex constraints are our main topic, an additional treatment of multiperiod or continuous models would by far exceed the scope of this thesis. Therefore, for an introduction and an overview of the different methodologies applied in multiperiod portfolio selection, the reader is referred to Steinbach [Ste01]. A good starting point for continuous models in general is a survey by Sundaresan [Sun00].
We assume further that the investment decision in the presence of uncertainty is based on the so-called *expected utility hypothesis*, which says that the optimal decision under uncertainty is the one that maximizes expected utility (cf. von Neumann and Morgenstern [vNM44]). The expected utility hypothesis is not without contentious points, as documented e.g. by Ellsberg [Ell61], Kahnemann and Tversky [KT79, KT04], or Rabin [Rab00]. Nevertheless, the hypothesis is accepted in many standard texts and will be presumed to be valid in the remainder of this thesis.
The Taylor-Expansion of the utility function $u(W_1)$ at position $E(W_1)$ results in the following equation:

$$\begin{aligned} u(W_1) &= u(E(W_1)) + u'(E(W_1))\,(W_1 - E(W_1)) \\ &\quad + \frac{1}{2}u''(E(W_1))(W_1 - E(W_1))^2 + \sum_{n=3}^{\infty}\frac{1}{n!}u^{(n)}(E(W_1))m^n(W_1) \end{aligned} \tag{2.1}$$

where $E(W_1)$ is the expected end-of-period wealth and $m^n(W_1)$ is the nth moment of W_1 at position $E(W_1)$.
When Eq. 2.1 is used to express the expected utility of the investor, we get the following result:

$$\begin{aligned} E(u(W_1)) &= E(u(E(W_1))) + E(u'(E(W_1))(W_1 - E(W_1))) \\ &\quad + E(\frac{1}{2}u''(E(W_1))(W_1 - E(W_1))^2) + E(\sum_{n=3}^{\infty}\frac{1}{n!}u^{(n)}(E(W_1))m^n(W_1)) \\ &= u(E(W_1)) + u'(E(W_1))E(W_1 - E(W_1)) \\ &\quad + \frac{1}{2}u''(E(W_1))E((W_1 - E(W_1))^2) + E(\sum_{n=3}^{\infty}\frac{1}{n!}u^{(n)}(E(W_1))m^n(W_1)) \\ &= u(E(W_1)) + u'(E(W_1))(E(W_1) - E(W_1)) + \frac{1}{2}u''(E(W_1))V(W_1) \\ &\quad + E(\sum_{n=3}^{\infty}\frac{1}{n!}u^{(n)}(E(W_1))m^n(W_1)) \\ &= u(E(W_1)) + \frac{1}{2}u''(E(W_1))V(W_1) + \underbrace{E(\sum_{n=3}^{\infty}\frac{1}{n!}u^{(n)}(E(W_1))m^n(W_1))}_{s} \end{aligned} \tag{2.2}$$

The main argument of Markowitz [Mar59, Mar87] is that if the utility function of the investor is quadratic or if it can be approximated with sufficient precision by a quadratic function, then the term s in Eq. 2.2 becomes 0. In this case, expected utility can be expressed solely in terms of expected return $E(W_1)$ and variance $V(W_1)$. If it is further assumed that the utility function is concave, i.e. the second derivative is negative, then from all portfolios with the same expected return the one with the smallest variance maximizes expected utility.
Concave utility functions that are quadratic have one serious drawback: For each one there is an input value above which the gradient of the utility function becomes negative. An increase in wealth would therefore decrease utility, which is not compatible with what would usually be seen as a rational behavior.
Markowitz [Mar59, Mar87] argues that for reasonable utility functions the quadratic approximation should be good enough in a sufficiently large area around $E(W_1)$ to prevent a major loss caused by approximation errors. Empirical results by Kallberg and Ziemba [KZ83] and Kroll et al. [KLM84] confirm this proposition.

If the utility function is not quadratic, there is a second reason why it makes sense to focus solely on expected return and variance. Under the condition of multivariate normally distributed asset returns, the return distribution of every portfolio consisting of those assets is also Gaussian, due to the fact that the normal distribution is stable. Moreover, any normal distribution is completely defined by its first and second moment (i.e. expected return and variance). Therefore, as long as the investor is risk averse, the conclusion is the same as above, irrespective of the type of utility function: for any given value of expected return, the portfolio with the smallest variance maximizes expected utility.
One obvious problem with using Gaussian distributions to model asset returns is based on the attribute of the normal distribution to be unbounded from below: if the investment alternatives are regular shares, their value can not fall below 0, i.e. there is not even the smallest probability for the return to be smaller than -1.
The main argument against normality of the asset returns is, however, that there is a lot of empirical evidence that investment returns are not multivariate Gaussian. Classical references documenting this are e.g. Mandelbrot [Man63] and Fama [Fam65].
But even if neither of the circumstances mentioned above (quadratic utility or normally distributed asset returns) are assumed to be true, there is a good chance that the portfolio that maximizes expected utility is fairly close to the one that minimizes variance for a given value of expected return (see e.g. Kroll et al. [KLM84] or Cremers et al. [CKP03]).
In the problematic case that expected utility has to be maximized with neither quadratic utility nor normally distributed returns, there is no other choice but to explicitly determine the utility function of the investor – which is often not an easy task – and then to directly use it in the optimization process. Depending on the type of the utility function, this may be nearly impossible to do in a reasonable amount of time.
For this reason, this thesis is restricted to mean-variance optimization.

2.2 The Standard Mean-Variance Model

The standard one-period mean-variance (MV) optimization problem can be expressed as a bicriteria optimization model where the solution simultaneously maximizes expected return and minimizes portfolio variance with respect to a given set of equality and inequality constraints:

Standard Mean-Variance Model (SMVM)

$$\min V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \tag{2.3a}$$

$$\max E(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{\mu} \tag{2.3b}$$

subject to

$$\boldsymbol{A_I x} \leq \boldsymbol{b_I} \tag{2.3c}$$

$$\boldsymbol{x}^T \boldsymbol{e} = 1 \tag{2.3d}$$

$$\boldsymbol{A_E x} = \boldsymbol{b_E} \tag{2.3e}$$

Element x_i of the vector $\boldsymbol{x} \in \mathbb{R}^n$ denotes the fraction of the budget invested in asset i. $\boldsymbol{C} \in \mathbb{R}^{n \times n}$ is the covariance matrix, $\boldsymbol{e} \in \mathbb{R}^n$ represents the unit vector, $\boldsymbol{\mu}$ is the vector of expected returns of all assets. $\boldsymbol{A_I}$ and $\boldsymbol{A_E}$ are the coefficients matrices of inequalities and equalities; $\boldsymbol{b_I}$ and $\boldsymbol{b_E}$ denote the corresponding right hand sides. Equation 2.3d (the budget constraint) guarantees that the fractions of the budget add up to 1. The budget constraint can be easily expressed as a part of the equations that are modeled by $\boldsymbol{A_E x} = \boldsymbol{b_E}$. We have mentioned it separately, however, as the constraint is often written down explicitly in other publications as well, probably due to its effect to normalize the solutions.

If the investor does not have to spend the complete budget, i.e., if he is allowed to keep a cash reserve (or if he can invest in a riskless asset), this can be easily integrated into the model by adding an asset with the desired yield (0 or a riskless interest rate) and a standard deviation of 0. Additionally, the "new" asset has to be uncorrelated to all other assets[1].

Other types of constraints compatible with the standard model but often mentioned separately are e.g. the prohibition of short sales, sector constraints, and upper bounds on asset weights. They are discussed in more detail in Section 2.5.

The necessary data for the mean-variance model consists of the expected return for every asset – an n-vector – and the respective $n \times n$ covariance matrix. Since the covariance matrix is symmetric, we require in total n variances and $n(n-1)/2$ covariances. In total we therefore need to acquire $2n + n(n-1)/2 = \frac{1}{2}n(n+3)$ data elements prior to the actual mean-variance optimization. To find a good estimate for that many numbers is of critical importance, since even small estimation errors can have grave consequences for results of the optimization. Kallberg and Ziemba [KZ83] and Chopra and Ziemba [CZ93] have found that mean-variance optimization is especially sensitive to variations of the

[1] If we assume that the riskless asset has the index k, this is achieved by setting all elements of the covariance matrix that have either row or column index k to 0.

expected returns. Best and Grauer [BG91] described the analytical framework to perform sensitivity analysis with respect to changes of the vector of expected returns and the right hand sides of the constraints. This framework is closely related to parametric quadratic programming algorithms that are discussed extensively in Chapter 3. A more general discussion of the difficulties to apply mean-variance analysis in practice can be found in Michaud [Mic89], although again the main focus is put on the sensitivity of the input data.
Several publications propose techniques to get better estimates of both expected returns and the covariance matrix. See e.g. Jobson and Korkie [JK80], Black and Litterman [BL92], Chopra et al. [CHT93], Ledoit and Wolf [LW04], Elton et al. [EGS06] and the references therein. Following a different approach, Jagannathan and Ma [JM03] propose to introduce nonnegativity constraints instead of more advanced parameter estimation techniques.
This thesis is concerned mainly with the actual optimization algorithms and not with the generation of the required input data. In the remainder of the thesis we therefore assume that the given data (the vector of expected returns and the covariance matrix) is correct.
There is usually no single portfolio that both minimizes variance and maximizes expected return. Instead, the result of an optimization based on the SMVM is generally a set of efficient portfolios.

Definition. A portfolio is *efficient / Pareto optimal* in the context of mean-variance portfolio selection if and only if there is no other feasible portfolio that improves at least one of the two optimization criteria without worsening the other.

When a portfolio is efficient, there is no other portfolio that complies with the constraints and has

1. lower variance and higher expected return or
2. lower variance and the same expected return or
3. the same variance and higher expected return.

The set of all efficient portfolios is called the *Pareto front*, *Pareto Frontier*, or the *Efficient Frontier*.
There are three well-established approaches to calculate a "solution" for problem SMVM: the ϵ-Constraint approach, the weighted sum method, and algorithms for parametric quadratic programming. Which of these is to be selected depends on the goal of the optimization, and on the capabilities of the software packages that are available for the task.
We will briefly discuss all three in this section[2].

[2] An extended presentation of the parametric quadratic programming approach can be found in Chapter 3, and the ϵ-Constraint method plays an important role in Chapter 4.

ϵ-Constraint Approach

If it is our intention to find the point on the Efficient Frontier with the minimum variance that has an expected return of at least E_f, this automatically removes one objective function and introduces an additional constraint. The resulting optimization problem is – as the covariance matrix is positive semidefinite – a convex quadratic programming problem (QP):

ϵ-Constrained Quadratic Programming Model (ϵ-QPM)

$$\min V(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} \tag{2.4a}$$

subject to

$$\boldsymbol{x}^T \boldsymbol{\mu} \geq E_f \tag{2.4b}$$

$$\boldsymbol{A_I x} \leq \boldsymbol{b_I} \tag{2.4c}$$

$$\boldsymbol{x}^T \boldsymbol{e} = 1 \tag{2.4d}$$

$$\boldsymbol{A_E x} = \boldsymbol{b_E} \tag{2.4e}$$

The solution of this model can be easily computed by using a QP-solver from one of several more advanced optimization software packages. A list of suitable programs and libraries is provided by the NEOS Guide [NEO06].
Such a solution, however, represents only one point on the Efficient Frontier. An *approximation* of the complete Pareto front can be computed by repeatedly solving the ϵ-QPM with increasing (or decreasing) E_f. In multicriteria optimization, this methodology is usually called *ϵ-Constraint method*. For further information on the ϵ-Constraint approach from a general multicriteria point of view, the reader is referred to Changkong and Haimes [CH83], or to Miettinen [Mie98].
One main drawback attributed to the ϵ-Constraint method is the time it requires to generate a sufficiently precise approximation, as the ϵ-QPM has to be solved for a large number of different values of E_f. Steuer et al. [SQH06] measured the time it took for only a very crude approximation (20 different values of E_f) with a commercial optimization package. Their conclusion was that for larger problem sizes, the slowness of the approach made this method inferior to parametric quadratic programming.
Unfortunately, the ϵ-Constraint method is the only approach most software packages and toolboxes offer for portfolio selection (for more details, see Steuer et al. [SQH06]).

Weighting Approach

The weighting method is another very basic but widely used approach in multicriteria optimization (Miettinen [Mie98]). In the field of portfolio selection, models of this type are regularly employed. Furthermore they form the basis for parametric quadratic programming algorithms. By default, a model based on the weighting methodology is similar to the ϵ-QPM insofar as its solution is also just a single point on the Pareto front.

Instead of turning one objective into an additional constraint, however, the "new" single objective function F is a weighted sum (or difference) of both objective functions from the SMVM:

λ_e**-QPM**

$$\min F(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{C}\boldsymbol{x} - \lambda_e\boldsymbol{x}^T\boldsymbol{\mu} \tag{2.5a}$$

subject to

$$\boldsymbol{A_I}\boldsymbol{x} \le \boldsymbol{b_I} \tag{2.5b}$$

$$\boldsymbol{x}^T\boldsymbol{e} = 1 \tag{2.5c}$$

$$\boldsymbol{A_E}\boldsymbol{x} = \boldsymbol{b_E} \tag{2.5d}$$

In order to approximate the Efficient Frontier, the λ_e-QPM has to be solved for different values of λ_e. It is sufficient to look at the solutions for $\lambda_e \ge 0$, as with them, all the points on the Efficient frontier can be calculated. The same procedures that solve the ϵ-QPM can be used here as well, as most quadratic programming solvers permit a linear term in the otherwise quadratic objective function.
The solution sets that can be calculated for varying parameters (either E_f or λ_e) are the same for both models: The Lagrange functions are identical if the parameter λ_e from model λ_e-QPM is interpreted as the multiplier for the expected return constraint in the ϵ-QPM[3]:

$$L(\boldsymbol{x}, \boldsymbol{\nu}, \boldsymbol{\lambda}, \boldsymbol{\lambda_e}) = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \lambda_e\,\boldsymbol{x}^T\boldsymbol{\mu} + \boldsymbol{\nu}^T(\boldsymbol{A_I}\boldsymbol{x} - \boldsymbol{b_I}) + \boldsymbol{\lambda}^T(\boldsymbol{A_E}\boldsymbol{x} - \boldsymbol{b_E})$$

As a consequence, the Karush-Kuhn-Tucker conditions – which are necessary and sufficient for optimality if the objective function and the constraints are convex – are identical as well:

$$\nabla L(\boldsymbol{x}, \boldsymbol{\nu}, \boldsymbol{\lambda}, \boldsymbol{\lambda_e}) = 0 \tag{2.6a}$$

$$\nu_i\,(\sum_{j=1}^{N} a_{ij}x_j - b_i) = 0 \qquad \forall\, i = 1, \ldots, l \tag{2.6b}$$

$$\boldsymbol{A_I}\boldsymbol{x} \le \boldsymbol{b_I} \tag{2.6c}$$

$$\boldsymbol{A_E}\boldsymbol{x} = \boldsymbol{b_E} \tag{2.6d}$$

$$\boldsymbol{\nu}, \lambda_e \ge 0 \tag{2.6e}$$

Each value of $\lambda_e \in [0; \infty)$ is mapped to exactly one value of E_f. For a proof and further details, the reader is referred to Markowitz [Mar87]. It is obvious that for $\lambda_e = 0$ the solution of the λ_e-QPM is the *Minimum Variance Portfolio (MVP)*, and that if λ_e is large enough, the calculated solution is the portfolio with maximum expected return.

[3] For sake of brevity, the budget constraint $\boldsymbol{x}^T\boldsymbol{e} = 1$ has been considered as part of the general equations.

Parametric Quadratic Programming Approach

If the Pareto front as a whole has to be calculated for a portfolio selection problem of type SMVM, the only choice is an active set algorithm for parametric quadratic programming (cf. Chapter 3). This algorithm solves the λ_e-QPM parametrically for all λ_e in the interval $[0, +\infty)$.
Starting from one point on the Efficient Frontier, the algorithm computes a sequence of so called *corner portfolios* $\boldsymbol{x_1}, \ldots, \boldsymbol{x_m}$. These corner portfolios define the complete Efficient Frontier as all other points on the Pareto front are convex combinations of the two adjacent corner portfolios:
If $\boldsymbol{x_i}$ and $\boldsymbol{x_{i+1}}$ are adjacent corner portfolios with expected returns E_i and E_{i+1}, $E_i \leq E_{i+1}$, then for every $E_{i,\lambda}$ with $E_{i,\lambda} = \lambda E_i + (1-\lambda)E_{i+1}$, $\lambda \in [0,1]$ the optimal portfolio $\boldsymbol{x_\lambda}$ is calculated as $\boldsymbol{x_\lambda} = \lambda \boldsymbol{x_i} + (1-\lambda)\boldsymbol{x_{i+1}}$.
Depending on the capabilities of the algorithm that is used to solve the mean-variance optimization problem, the linear constraints may have to be adapted to the required formulation. The "classical" parametric quadratic programming algorithm from Markowitz [Mar87], the *Critical Line Algorithm*, supports only equations and nonnegativity constraints, i.e. inequalities which ensure that variables remain positive. Therefore, general inequalities of the type $\boldsymbol{Ax} \leq \boldsymbol{b}$ have to be transformed into equations by using slack variables (see, e.g., Markowitz [Mar87] and especially Rudolf [Rud94]). This approach is problematic as it increases the problem size due to the additional variables.
In Chapter 3 we present an optimized version of a parametric quadratic programming algorithm that accepts both equations and inequalities. Thus, no modifications to the problem structure are necessary[4].

An important simplification common to all portfolio selection models mentioned above is that the elements of $\boldsymbol{x}$ are assumed to be real numbers. Considering that shares can usually not be bought and sold in fractions, this may have the effect that the solution of the SMVM (and therefore also solutions of the ϵ-QPM and the λ_e-QPM) may not be applicable to the actual optimization problem of the investor. The divergence can be quite significant if the available budget is small. Given a larger budget, however, the difference between the solution of problem SMVM and the solution of the actual optimization problem – where the number of traded assets has to be an integer – is negligible[5].
The computational difficulties that result if integer constraints are included in the optimization are briefly discussed in Section 2.5.2 together with problems caused by other types of constraints that can not be integrated into the standard model.

[4]Naturally, all variables have to be shifted to the left hand side of the constraints, and "larger than" inequalities have to be multiplied by -1.

[5]Another justification for using real valued variables is that in several interviews, portfolio managers for mutual funds did confirm that they only work with fractions of the available budget due to the fact that their budget is subject to daily changes.

2.3 Measures of Risk

One common interpretation of the variance in the standard model is that it quantifies portfolio risk. Therefore, mean-variance optimization is often referred to as risk-return optimization. This point of view is controversial as the meaning of the term *risk* in everyday perception clashes with the mathematical definition of variance. One main problem of variance as a measure of risk is that both positive and negative deviations of the actual return from the expected portfolio return are equally taken into account when the variance is calculated. Only very few investors will, however, consider it a problem if the portfolio return is larger than the return that was expected before. A risk measure that only measures the downside deviations while leaving out the upside potential may be more compatible with what would usually be expected from a measure of risk.
As a consequence of this problem and also due to the rising importance of risk management in financial institutions, which are also mainly concerned with negative deviations of the return, several authors have examined the application of alternative risk measures in portfolio selection with the intention to better capture the notion of risk.
Two approaches that have played a prominent role as risk measures in the last few years are Value at Risk (VaR) and Conditional Value at Risk (CVaR).
Value at Risk is a concept that describes risk as the loss of a portfolio of assets which is not surpassed given a confidence level α. The VaR is therefore the difference between the expected return of the portfolio and the $(1-\alpha)$-quantile of its return distribution. Due to certain drawbacks of the VaR-approach like, e.g., the lack of sub-additivity (see Artzner et al.[ADEH99]), Conditional Value at Risk is often suggested as a suitable replacement. Conditional Value at Risk (CVaR), also called Expected Tail Loss (ETL), is the expected loss under the condition that the portfolio return is below the same α-quantile that marks the threshold of the VaR.
There are other measures of risk that try to capture the asymmetric meaning of risk, with the semivariance measure suggested by Markowitz [Mar59] being the most prominent. Grootveld and Hallerbach [GH99] analyze different downside-risk measures and compare the results of their application to those of the standard mean-variance framework.

Konno and Yamazaki [KY91] proposed to replace portfolio variance with the so-called Mean Absolute Deviation (MAD) which is defined as follows:

$$\omega_p = E(\mid \boldsymbol{x}'\boldsymbol{r} - \boldsymbol{\mu}'\boldsymbol{x} \mid)$$

with $\boldsymbol{r}$ as the vector of random variables representing the returns of all assets, $\boldsymbol{x}$ as vector of portfolio weights and $\boldsymbol{\mu}$ as vector of expected returns. Their main arguments for this modification were:

1. With MAD, no covariance matrix is necessary. Therefore the number of parameters to be estimated before the optimization is significantly lower.

2. Konno and Yamazaki claim that quadratic mean-variance optimization with large dense covariance matrices is computationally not feasible. In the MAD model, it is only necessary to solve a linear optimization problem.

3. They also argue that the solution of their optimization model results in fewer assets being included in the portfolio whereas for the mean-variance case the number of assets with a weight larger than 0 may be large.

The second reason does not align with our experience (cf. Chapter 3), and we also did not witness the effect that the mean-variance model results in a large number of small weighted assets. Simaan [Sim97] compared both models with respect to the consequences of estimation errors in the parameters. He concluded that the resulting error is less severe in the mean-variance model.
In this thesis, neither VaR, nor CVaR, nor MAD will be considered any further. Instead, we will focus on the variance of the portfolio return. For additional information on VaR and CVaR in the context of portfolio selection, the reader is referred to e.g. Uryasev [Ury00], Krokhmal et al. [KPU02], Maringer [Mar05], Gaivoronsky and Pflug [GP05], or Alexander and Baptista [AB04].
Besides the initial paper from Konno and Yamazaki [KY91], the MAD model is discussed and extended in Konno and Wijayanayake [KW02] and different solution approaches are treated in Konno and Yamamoto [KY05]. Mansini et al. [MOS03] give an overview of the different LP-solvable portfolio optimization problems, among them the MAD and the CVaR model. They also provide a computational comparison of the different models on real life data.

2.4 Benchmark Problems

Many authors test their approaches on the publicly available benchmarks provided in the OR-library [Bea06]. For mean-variance portfolio selection, 5 data sets are available, which we will use as well, namely

- P1 The smallest problem, the Hang Seng benchmark consisting of 31 assets.
- P2 The benchmark data set based on the DAX 100 containing 85 assets.
- P3 The benchmark based on the FTSE 100 with 89 assets.
- P4 The S&P benchmark with 98 assets.
- P5 The largest problem in the OR-library, the Nikkei 225 benchmark with 225 assets.

The data sets consist of values for the expected return and the standard deviation of each asset and of the correlation matrix. They were initially used by Chang et al. [CMBS00]. Since we also required larger data sets in order to examine which approaches scale well, we additionally tested it on larger problem instances:

- P6 A benchmark with 500 assets
- P7 A benchmark with 1000 assets
- P8 A benchmark with 2000 assets

The data sets were generously provided by the authors of Hirschberger et al. [HQS07]. They were generated according to a method described in that paper.